

ADE SUBALGEBRAS OF THE TRIPLET VERTEX ALGEBRA $\mathcal{W}(p)$: A_m -SERIES

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ABSTRACT. Motivated by [5], for every finite subgroup $\Gamma \subset PSL(2, \mathbb{C})$ we investigate the fixed point subalgebra $\mathcal{W}(p)^\Gamma$ of the triplet vertex $\mathcal{W}(p)$, of central charge $1 - \frac{6(p-1)^2}{p}$, $p \geq 2$. This part deals with the A -series in the ADE classification of finite subgroups of $PSL(2, \mathbb{C})$. First, we prove the C_2 -cofiniteness of the A_m -fixed subalgebra $\mathcal{W}(p)^{A_m}$. Then we construct a large family of $\mathcal{W}(p)^{A_m}$ -modules, which are expected to be a complete set of irreps. As a strong support to our conjecture we prove modular invariance of (generalized) characters of the relevant (logarithmic) modules. Further evidence is provided by calculations in Zhu's algebra for $m = 2$. We also present a rigorous proof of the fact that the full automorphism group of $\mathcal{W}(p)$ is $PSL(2, \mathbb{C})$.

1. INTRODUCTION

Some of the most important examples of rational vertex algebras come from lattice vertex algebras (e.g. Moonshine Module). More recently, many interesting examples of W -algebras have been constructed either as subalgebras of lattice vertex algebras or as their cosets or orbifolds. For example, rational vertex algebras in the conjectural $c = 1$ classification should all arise in this way (cf. [12] [10]). When going beyond unitary theories (easily achieved by deforming the canonical quadratic Virasoro vector) the structure of lattice vertex algebra changes dramatically when viewed as a Virasoro algebra module, and many "symmetries" (i.e. automorphisms) are broken. Still, for certain values of central charge such lattice vertex algebras can have large interesting subalgebras. Triplet vertex algebra $\mathcal{W}(p)$ is one important example; it is constructed as a subalgebra of the rank one lattice vertex algebra with central charge $1 - \frac{6(p-1)^2}{p}$, by taking the kernel of the short screening. It is no longer rational, but as shown in [5], it is C_2 -cofinite. We expect that C_2 -property persists in the higher rank as well [8] and for other vertex algebras of triplet-type [7].

At the end of [5], motivated by important works [14] [15] [16], it was pointed out that $\mathcal{W}(p)$ admits a hidden action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ via derivations. Although this claim has not been proven, it is often assumed to be true due to considerations of quantum groups (cf [15]). Here

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we first present a proof of this claim for completeness¹ (cf. Theorem 2.3 and Appendix). Our proof is based on the results from [5] and on an explicit construction of an automorphism Ψ of order two. Because our Lie algebra acts via derivations, its generators can exponentiate to $PSL(2, \mathbb{C})$ as a group of automorphisms of $\mathcal{W}(p)$. Then, as we already mentioned in [5], to each finite subgroup Γ of $PSL(2, \mathbb{C})$, we can now associate the orbifold subalgebra $\mathcal{W}(p)^\Gamma$. The main purpose of this series of papers is to study the C_2 -cofiniteness of $\mathcal{W}(p)^\Gamma$ and their representations. The finite subgroups of $PSL(2, \mathbb{C})$ are well known to follow the ADE classification (cf. [13]). Up to conjugation, these are cyclic groups, dihedral groups, and three exceptional finite subgroups (tetrahedral, octahedral and icosahedral groups). Motivated by the the rational $c = 1$ case [9], [10] we expect that

Conjecture 1.1. *For each Γ in the ADE classification, $\mathcal{W}(p)^\Gamma$ is C_2 -cofinite.*

This paper deals primarily with the case when $\Gamma = A_m$, that is, the automorphism group is a finite cyclic group of order m . This case is easier to handle due to the fact that the orbifold algebra $\mathcal{W}(p)^\Gamma$ can be defined as a subalgebra of the orbifold subalgebra V_L^Γ , so no $\mathfrak{sl}_2(\mathbb{C})$ considerations are needed. After we recall several structural results on the triplet vertex algebra and its orbifold $\mathcal{W}(p)^{A_m}$ we prove

Theorem 1.2. *For Γ of type A_m , the corresponding invariant subalgebra $\mathcal{W}(p)^\Gamma$ is C_2 -cofinite.*

After this we move on to study structure of irreducible $\mathcal{W}(p)^{A_m}$ -modules. Irreducible modules come in three categories: Λ , Π and R -series, coming from the $\mathcal{W}(p)$ -modules of type Λ and Π and twisted V_L -modules, respectively. We prove the following result

Theorem 1.3. *Three series of irreducible modules contribute with $2pm^2$ irreducible $\mathcal{W}(p)^{A_m}$ -modules. Conjecturally, these are all irreducible modules, up to isomorphism.*

Next, we consider the $m = 2$ case via Zhu's algebra $A(\mathcal{W}(p)^{A_m})$. We give further evidence for the above conjecture based on the properties of $A(\mathcal{W}(p)^{A_2})$.

Finally, in the last section, we are concerned with graded characters of $\mathcal{W}(p)^{A_m}$ -modules and their modular closure. As in [5], we show that the space of characters close under $SL(2, \mathbb{Z})$ if we include certain generalized characters coming from logarithmic modules. Our result in this direction is

Theorem 1.4. *The vector space generated by the characters and generalized characters coming from these three series is $pm^2 + 2p - 1$ -dimensional. Moreover, compared to $\mathcal{W}(p)$, no new generalized characters occur in $\mathcal{W}(p)^{A_m}$.*

This modular invariance result is a strong evidence that our classification of irreps is complete.

¹Of course, there is no action of sl_2 on all of V_L , for $p \geq 2$.

We finish this introduction with a few words about a possible Kazhdan-Lusztig dual quantum group of $\mathcal{W}(p)^\Gamma$. It is known that the abelian category $\overline{U}_q(sl_2)\text{-Mod}$ is equivalent to the category $\mathcal{W}(p) - \text{Mod}$ [?]. Therefore it seems natural to have an embedding

$$\overline{U}_q(sl_2) \hookrightarrow \overline{U}_q(sl_2)^\Gamma,$$

where $\overline{U}_q(sl_2)^\Gamma$ is yet-to-be defined finite-dimensional quantum group, such that $\mathcal{W}(p)^\Gamma\text{-Mod}$ and $\overline{U}_q(sl_2)^\Gamma\text{-Mod}$ are equivalent (as abelian categories). What is puzzling to us at this stage, is the meaning of the ADE classification on the quantum group side.

2. THE TRIPLET VERTEX ALGEBRA $\mathcal{W}(p)$ AS A $\mathfrak{sl}_2(\mathbb{C})$ -MODULE

We begin with some preliminaries about the triplet vertex algebra $\mathcal{W}(p)$; more details can be find in [5]. Fix an integer $p \geq 2$. Let $L = \mathbb{Z}\alpha$ be a rank one lattice with a bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle \alpha, \alpha \rangle = 2p,$$

Let

$$V_L = \mathcal{U}(\widehat{\mathfrak{h}}_{<0}) \otimes \mathbb{C}[L]$$

be the corresponding lattice vertex operator algebra [22], where $\widehat{\mathfrak{h}}$ is the affinization of $\mathfrak{h} = \mathbb{C}\alpha$ induced by the bilinear form, and $\mathbb{C}[L]$ is the group algebra of L . In V_L we choose the following conformal vector:

$$\omega = \frac{\alpha(-1)^2}{4p} \mathbf{1} + \frac{p-1}{2p} \alpha(-2) \mathbf{1}.$$

Thus V_L is a Virasoro algebra module of central charge

$$c_p = 1 - \frac{6(p-1)^2}{p}.$$

We recall the following two results [3, 4, 5]:

Lemma 2.1. *let $Q = e_0^\alpha : V_L \rightarrow V_L$ be the screening operator and let $n \in \mathbb{Z}_{\geq 0}$, then $Q^i e^{-n\alpha} \neq 0$ if and only if $i \leq 2n$.*

Set $u_n^i = Q^i e^{-n\alpha}$, $i, n \in \mathbb{Z}_{\geq 0}$, $i \leq 2n$.

Lemma 2.2. *Each u_n^i is a singular vector of V_L (as a Virasoro algebra module). The submodule generated by these u_n^i is isomorphic to*

$$\overline{V}_L \cong \bigoplus_{n=0}^{\infty} (2n+1)L(c_p, n^2p + np - n).$$

Moreover \overline{V}_L is a subalgebra of V_L .

Set the triplet vertex algebra

$$\mathcal{W}(p) = \overline{V}_L.$$

Let V be a vertex operator algebra and let $C_2(V)$ be the linear span of elements of type $a_{-2}b$, $a, b \in V$. Then $V/C_2(V)$ has a commutative algebra structure with the multiplication

$$\bar{a} \cdot \bar{b} = \overline{a_{-1}b}$$

where $-$ is the projection from V to $V/C_2(V)$. V is said to be C_2 -cofinite if $V/C_2(V)$ is finite-dimensional.

It was proved in [5] that $\mathcal{W}(p)$ is C_2 -cofinite but irrational. Using the C_2 -cofiniteness of $\mathcal{W}(p)$ and a result of [20], the first two authors proved the existence of logarithmic $\mathcal{W}(p)$ -modules in [5, 6].

In the end of [5], it was announced out that $\mathcal{W}(p)$ admits a hidden action of \mathfrak{sl}_2 as follows: Set $e = Q$, $h = \frac{\alpha(0)}{p}$. Let $f \in \text{End}_{\text{Vir}}(\mathcal{W}(p))$ be the unique operator defined by

$$fe^{-n\alpha} = 0, \quad fQ^ie^{-n\alpha} = -i(i-2n-1)Q^{i-1}e^{-n\alpha}, \quad 1 \leq i \leq 2n.$$

One checks that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Thus we have

$$\mathcal{W}(p) = \bigoplus_{n=0}^{\infty} W_{2n+1} \otimes L(c_p, n^2p + np - n)$$

where W_{2n+1} is a $(2n+1)$ -dimensional irreducible \mathfrak{sl}_2 -module. We extend the action of e, f, h by letting them commute with the Virasoro generators.

It is clear that e and h are derivation on the vertex operator algebra $\mathcal{W}(p)$. We shall prove in Appendix that there exists an automorphism $\Phi \in \text{Aut}(\mathcal{W}(p))$ of order two of $\mathcal{W}(p)$ that satisfies

$$\Psi(Q^ie^{-n\alpha}) = \frac{(-1)^i i!}{(2n-i)!} Q^{2n-i} e^{-n\alpha}.$$

Then the operator f can be represented as

$$f = -\Psi^{-1}Q\Psi.$$

Therefore f is also a derivation on $\mathcal{W}(p)$. Thus the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on $\mathcal{W}(p)$ by derivations; i.e.

$$(1) \quad x(a_nb) = (xa)_nb + a_n(xb)$$

for all $n \in \mathbb{Z}$, and $a, b \in \mathcal{W}(p)$ and $x \in \mathfrak{sl}_2$. This property is known to hold for $p = 2$ by using explicit expression for f . The integration of the action of \mathfrak{sl}_2 gives an action of $PSL(2, \mathbb{C})$ on the vertex operator algebra $\mathcal{W}(p)$ as an automorphism group [5] (see [9] for the $c = 1$ case).

Theorem 2.3. *The group $PSL(2, \mathbb{C})$ acts on $\mathcal{W}(p)$ as an automorphism group. Moreover,*

$$\text{Aut}(\mathcal{W}(p)) \cong PSL(2, \mathbb{C}).$$

Proof. By the above arguments, we know that $PSL(2, \mathbb{C})$ is a subgroup of the full automorphism group $\text{Aut}(\mathcal{W}(p))$.

In order to prove the converse, it suffices to construct homomorphism $\text{Aut}(\mathcal{W}(p)) \rightarrow PSL(2, \mathbb{C})$. Define

$$e^1 = \frac{U^+}{2i}, \quad e^2 = \frac{U^-}{2}, \quad e^3 = H,$$

where $U^\pm = 2F \pm E$.

Let Ψ be an automorphism of $\mathcal{W}(p)$. It defines an matrix $[\Psi] = (a_{i,j})$ such that

$$\Psi(e^i) = a_{i,1}e^1 + a_{i,2}e^2 + a_{i,3}e^3.$$

One can easy see that $[\Psi] \in SO(3, \mathbb{C})$. This gives a group homomorphism:

$$G : \text{Aut}(\mathcal{W}(p)) \rightarrow PSL(2, \mathbb{C}) \cong SO(3, \mathbb{C}).$$

□

Remark 2.4. *It is useful to point out that the automorphism group of V_L , with ω as earlier, is much smaller generated by the automorphisms $\exp(\lambda\alpha(0))$.*

Up to conjugation, there are five classes of finite subgroups of $PSL(2, \mathbb{C})$ (cf.[13]). As there is a natural projection $PSL(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C})$, each subgroup Γ of $PSL(2, \mathbb{C})$ can be lifted to the double covering group $\bar{\Gamma}$ in $SL(2, \mathbb{C})$. In the following table we describe a set of generators for each finite subgroup of $PSL(2, \mathbb{C})$ in terms of elements of $SL(2, \mathbb{C})$. Let $V = \mathbb{C}^2$ be the

name	Γ	order	$\bar{\Gamma}$	generators
cyclic group	A_m	m	\bar{A}_m	$\begin{pmatrix} e^{\frac{\pi i}{m}} & 0 \\ 0 & e^{-\frac{\pi i}{m}} \end{pmatrix}$
dihedral group	D_m	$2m$	\bar{D}_m	$\begin{pmatrix} e^{\frac{\pi i}{m}} & 0 \\ 0 & e^{-\frac{\pi i}{m}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
tetrahedral group	T	12	\bar{T}	$\begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \frac{\sqrt{3}}{3} \begin{pmatrix} i & -1-i \\ 1-i & -i \end{pmatrix}$
octahedron group	O	24	\bar{O}	$\begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}, \sqrt{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$
icosahedron group	I	60	\bar{I}	$\begin{pmatrix} e^{\frac{\pi i}{10}} & 0 \\ 0 & e^{-\frac{\pi i}{10}} \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} \epsilon - \epsilon^4 & \epsilon^2 - \epsilon^3 \\ \epsilon^2 - \epsilon^3 & -\epsilon + \epsilon^4 \end{pmatrix}$

TABLE 1. Finite subgroups of $PSL(2, \mathbb{C})$, ($\epsilon = \frac{\pi i}{5}$)

standard \mathfrak{sl}_2 -module. Then the $2n$ -th symmetric power of V is isomorphic to W_{2n+1} by identifying $x^i y^{2n-i}$ with $(2n-i)! Q^i e^{-n\alpha}$. Under this identification, for each $\alpha \in PSL(2, \mathbb{C})$,

$$(2) \quad \alpha Q^i e^{-n\alpha} = \frac{1}{(2n-i)!} (ax + by)^i (cx + dy)^{2n-i}.$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the lifting of α in $SL(2, \mathbb{C})$. In the Appendix we will use Formula (2) to determine a set of generators for each $\mathcal{W}(p)^\Gamma$.

3. THE ALGEBRA $\mathcal{W}(p)^{A_m}$

Consider the automorphism $\sigma = \exp(\pi i \frac{\alpha(0)}{pm})$ acting on V_L , with $\langle \alpha, \alpha \rangle = 2p$. Denote by V_L^σ , the σ -fixed subalgebra of V_L . It is clear that

$$V_L^\sigma = \bigoplus_{n \in \mathbb{Z}} M(1) \otimes e^{nm\alpha}.$$

Now we have

$$(3) \quad \mathcal{W}(p)^{A_m} = \mathcal{W}(p) \cap V_L^\sigma.$$

It is relatively easy to see that the invariant subalgebra $\mathcal{W}(p)^{A_m}$, as a Virasoro algebra module, is generated by

$$u_n^i = Q^i e^{-n\alpha}, \quad i, n \in \mathbb{Z}_{\geq 0}, \quad i \leq 2n, \quad m|(n-i).$$

Set

$$\begin{aligned} F^{(m)} &= e^{-m\alpha}, \\ E^{(m)} &= Q^{2m} e^{-m\alpha}, \\ H &= Q e^{-\alpha}. \end{aligned}$$

As in the proof of Proposition 1.3 in [5], we quickly obtain the following result.

Theorem 3.1. *The vertex operator algebra $\mathcal{W}(p)^{A_m}$ is strongly generated by $E^{(m)}, F^{(m)}, H$ and ω .*

For a vertex algebra V , let $\mathcal{P}(V) = V/C_2(V)$ denotes the corresponding Zhu's Poisson C_2 -algebra. Let $M(1) = \hat{\mathfrak{h}}_{<0} \subset V_L$ be the vertex operator algebra associated to $\hat{\mathfrak{h}}$. Let $\overline{M(1)} = \mathcal{W}(p) \cap M(1)$. We need the following result

Lemma 3.2. *$\mathcal{P}(M(1))$ is the polynomial algebra generated by $\beta = \overline{\alpha(-1)\mathbf{1}}$. Let $C_p = \frac{(4p)^{2p-1}}{(2p-1)!^2}$, then*

$$\mathcal{P}(\overline{M(1)}) \cong \mathbb{C}[x, y]/(y^2 - C_p x^{2p-1})$$

with the isomorphism sending \overline{H} (resp. ω) to y (resp. x). Moreover, the inclusion $\overline{M(1)} \hookrightarrow M(1)$ induces an injective homomorphism

$$\varphi : \mathcal{P}(\overline{M(1)}) \rightarrow \mathcal{P}(M(1)).$$

such that $\varphi(\overline{\omega}) = \frac{1}{2}\beta^2$ and $\varphi(\overline{H}) = \frac{1}{(2p-1)!}\beta^{2p-1}$.

Proof. The structure of $\mathcal{P}(\overline{M(1)})$ can be easily deduced from the structure of Zhu's algebra $A(\overline{M(1)})$ (cf. [3]). The other statements are trivial. \square

Lemma 3.3. *$\overline{F_{-n}^{(m)} E^{(m)}} \neq 0$ in $\mathcal{P}(\overline{M(1)})$ for some $n > 2$.*

Proof. Let $\Delta_s(x_1, \dots, x_s) = \prod_{i < j} (x_i - x_j)$ be the Vandermonde determinant. For any $h \in \widehat{b}$ let

$$E^\pm(h, x_1, \dots, x_s) = \exp\left(\sum_{n \in \pm\mathbb{Z}_+} \frac{h(n)}{n} (x_1^{-n} + \dots + x_s^{-n})\right).$$

In the following we abbreviate $E^\pm(h, x_1, \dots, x_s)$ to $E^\pm(h, s)$, $x_1 \cdots x_{2m}$ to $\otimes x$, and $x_1 + \dots + x_{2m}$ to $\oplus x$. Direct calculations show that

$$\begin{aligned} & Y(F^{(m)}, x_0) E^{(m)} \\ &= \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} (\otimes x)^{-2mp} \Delta_{2m}^{2p} Y(E, x_0) E^-(-\alpha, 2m) e^{m\alpha} \\ &= \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} x_0^{-2m^2 p} (\otimes x)^{-2mp} \Delta_{2m}^{2p} E^-(m\alpha, x_0) E^+(m\alpha, x_0) E^-(-\alpha, 2m) \mathbf{1} \\ &= \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} x_0^{-2m^2 p} (\otimes x)^{-2mp} \Delta_{2m}^{2p} \left(\prod_{i=1}^{2m} \left(1 - \frac{x_i}{x_0}\right)^{-2mp} \right) E^-(m\alpha, x_0) E^-(-\alpha, 2m) \mathbf{1}. \end{aligned}$$

Assume the lemma is false. Then under the projection

$$M(1) \rightarrow M(1)/C_2(M(1)) = \mathbb{C}[\beta],$$

$$\begin{aligned} & \overline{Y(F^{(m)}, x_0) E^{(m)}} \\ &= \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} x_0^{-2m^2 p} (\otimes x)^{-2mp} \Delta_{2m}^{2p} \left(\prod (1 - \frac{x_i}{x_0})^{-2mp} \right) e^{\beta(\oplus x - mx_0)} \\ &= \beta^{2m(mp+p-1)} f(\beta x_0) \end{aligned}$$

for some Laurent polynomial f .

Set $\beta = 1$ in the above formula we get

$$\begin{aligned} & e^{-mx_0} \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} x_0^{-2m^2 p} (\otimes x)^{-2mp} \Delta_{2m}^{2p} \left(\prod (1 - \frac{x_i}{x_0})^{-2mp} \right) e^{(\oplus x)} \\ &= f(x_0). \end{aligned}$$

It is easy to see that

$$g(x_0) = \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} (\otimes x)^{-2mp} \Delta_{2m}^{2p} \left(\prod (1 - \frac{x_i}{x_0})^{-2mp} \right) e^{(\oplus x)}$$

is also a Laurent polynomial. Then the above formula forces $g(x_0) = 0$, otherwise

$$e^{-mx_0} x_0^{-2m^2 p} g(x_0)$$

will not be a Laurent polynomial. But the Morris constant term identity [21] implies that the coefficient of x_0^{-p+1} in $g(x_0)$ is

$$\begin{aligned} & \text{Res}_{x_1} \cdots \text{Res}_{x_{2m}} (x_1 \cdots x_{2m})^{-2mp} \Delta_{2m}^{2p} \left(\prod (1 - x_i)^{-2mp} \right) \\ &= \prod_{i=0}^{2m-1} \binom{(-2m+i)p}{p-1} \frac{(p-1)!((i+1)p)!}{(p-1+ip)!p!} \neq 0. \end{aligned}$$

This contradicts our assumption and completes the proof. \square

Theorem 3.4. $\mathcal{W}(p)^{A_m}$ is C_2 -cofinite.

Proof. As in the proof of Theorem 2.1 in [5], we deduce from Theorem 3.1 that the commutative algebra $\mathcal{W}(p)^{A_m}/C_2(\mathcal{W}(p)^{A_m})$ is generated by $\overline{E^{(m)}}, \overline{F^{(m)}}, \overline{H}$ and $\overline{\omega}$. Moreover

$$\overline{E^{(m)}}^2 = \overline{F^{(m)}}^2 = 0$$

and

$$\overline{H}^2 = C_p \overline{\omega}^{2p-1}.$$

According to Lemma 3.3, there exists some integer n , satisfying $n \geq 2$ and $\overline{F_{-n}^{(m)} E^{(m)}} \neq 0$ in $M(1)/C_2(M(1))$. This implies that

$$\overline{F_{-n}^{(m)} E^{(m)}} = a \overline{\omega^k H} \quad (k = m^2 p + mp - m - 2 + n/2, \ a \in \mathbb{C}, a \neq 0)$$

if n is even, or

$$\overline{F_{-n}^{(m)} E^{(m)}} = a \overline{\omega^k} \quad (k = m^2 p + mp - m + (n-1)/2, \ a \in \mathbb{C}, a \neq 0)$$

if n is odd.

On the other hand, we have $\overline{F_{-n}^{(m)} E^{(m)}} = 0$ in $\mathcal{W}(p)^{A_m}/C_2(\mathcal{W}(p)^{A_m})$. This implies that $\overline{\omega}, \overline{H}$ are also nilpotent in $\mathcal{P}(\mathcal{W}(p)^{A_m})$.

Therefore, $\mathcal{W}(p)^{A_m}/C_2(\mathcal{W}(p)^{A_m})$ is finite-dimensional. \square

4. TOWARDS IRREDUCIBLE $\mathcal{W}(p)^{A_m}$ -MODULES

There are several clues leading to classification of irreducible modules for orbifold vertex algebras. Folklore meta-theorem in this direction says that all V^G -modules should come from V -modules (by restrictions) and from g -twisted V -modules, where $g \in G$. This can be made more precise if the category of V -modules is a MTC. Guided by this principle, we now investigate irreducible \mathcal{W} -modules. In this part we classify such irreducible modules. Conjecturally, these are all irreducible modules. Support for that is presented in next chapters.

4.1. Irreducible $\mathcal{W}(p)^{A_m}$ -modules: Λ -series. In this part we construct pm irreducible $\mathcal{W}(p)^{A_m}$ -modules starting from $\mathcal{W}(p)$ -modules $\Lambda(1), \dots, \Lambda(p)$ (we use notation from [5]). Recall that $\Lambda(i)$ has lowest conformal weight

$$\deg(e^{(i-1)\alpha/2p}) = \frac{(i-1)(i-1-2p+2)}{4p} = h_{i,1}, \quad i = 1, \dots, p.$$

These modules are denoted by $\Lambda(1), \dots, \Lambda(p)$ in [5]. The next decomposition is well-known (cf. [5]):

$$\Lambda(i) = \bigoplus_{n=0}^{\infty} (2n+1) L(c_{p,1}, h_{i,2n+1}).$$

where the null vectors are given by $Q^j e^{-n+(i-1)\alpha/2p}$, in the natural range.

Let now $m = 2k$ (even).

Consider now for $j = 1, \dots, k-1$.

$$\begin{aligned}\Lambda(i)_0 &= \mathcal{W}(p)^{A_m} \cdot e^{(i-1)\alpha/2p} \\ \Lambda(i)_j^- &= \mathcal{W}(p)^{A_m} \cdot e^{-j\alpha + \frac{(i-1)\alpha}{2p}}, \quad j = 1, \dots, k-1 \\ \Lambda(i)_j^+ &= \mathcal{W}(p)^{A_m} \cdot Q^{2j} e^{-j\alpha + \frac{(i-1)\alpha}{2p}}, \quad j = 1, \dots, k-1 \\ \Lambda(i)_m &= \mathcal{W}(p)^{A_m} \cdot e^{-m\alpha/2 + \frac{(i-1)\alpha}{2p}}\end{aligned}$$

The next result comes immediately from the above decomposition

Proposition 4.1. *Let $m = 2k$. As Virasoro modules*

(1)

$$\Lambda(i)_0 = \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=0}^{m-1} L(c_{p,1}, h_{i,2(nm+k)+1}).$$

(2)

$$\begin{aligned}\Lambda(i)_j^- &= \Lambda(i)_j^+ = \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=j}^{m-j-1} L(c_{p,1}, h_{i,2(nm+k)+1}) \\ &\quad + \bigoplus_{n=0}^{\infty} (2n+2) \bigoplus_{k=m-j}^{m+j-1} L(c_{p,1}, h_{i,2(nm+k)+1}).\end{aligned}$$

(3)

$$\Lambda(i)_m = \bigoplus_{n=0}^{\infty} (2n+2) \bigoplus_{k=0}^{m-1} L(c_{p,1}, h_{i,2(nm+\frac{m}{2}+k)+1}).$$

For $m = 2k+1$ (odd case) the module $\Lambda(i)_m$ does not appear in the decomposition. Instead we have $\Lambda(i)_0$ and $\Lambda(i)_j^{\pm}$, $j = 1, \dots, k$ defined by the same formulas as above.

Proposition 4.2. *For $m = 2k+1 \geq 1$, we have*

(1)

$$\Lambda(i)_0 = \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=0}^{m-1} L(c_{p,1}, h_{i,2(nm+k)+1}).$$

(2)

$$\begin{aligned}\Lambda(i)_j^- &= \Lambda(i)_j^+ = \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=j}^{m-j-1} L(c_{p,1}, h_{i,2(nm+k)+1}) \\ &\quad + \bigoplus_{n=0}^{\infty} (2n+2) \bigoplus_{k=m-j}^{m+j-1} L(c_{p,1}, h_{i,2(nm+k)+1}).\end{aligned}$$

As in [5] we infer

Theorem 4.3. $\Lambda(i)_j^\pm$ are irreducible $\mathcal{W}(p)^{A_m}$ -modules. In particular, the vertex algebra $\mathcal{W}(p)^{A_m}$ is simple.

Proof. We first recall that $\Lambda(i)$ is an irreducible $\mathcal{W}(p)$ -modules. We have eigenspace decomposition

$$\Lambda(i) = \oplus_{j, \epsilon \in \pm} \Lambda(i)_j^\epsilon$$

with respect to the action of the automorphism σ . In particular we have a decomposition of $\Lambda(1) = \mathcal{W}(p)$. Fix $0 \neq v \in \Lambda(i)_j^\epsilon$. Then we have

$$\Lambda(i) = \text{Span}\{w_n v : n \in \mathbb{Z}, w \in \mathcal{W}(p)\}.$$

But eigenvalue decomposition implies

$$\Lambda(i)_j^\epsilon = \text{Span}\{w_n v : n \in \mathbb{Z}, w \in \mathcal{W}(p)^{A_m}\},$$

meaning that $\Lambda(i)_j^\epsilon$ are irreducible. \square

4.2. Irreducible $\mathcal{W}(p)^{A_m}$ -modules: Π -series. Recall $i \in \{1, \dots, p\}$,

$$\Pi(i) = \mathcal{W}(p) \cdot e^{\frac{-p-1+i}{2p}\alpha},$$

and

$$\Pi(i) = \bigoplus_{n=1}^{\infty} (2n) L(c_{p,1}, h_{i+p, 2n+1}).$$

Top component of $\Pi(i)$ is spanned by $e^{\frac{-p-1+i}{2p}\alpha}$ and $Qe^{\frac{-p-1+i}{2p}\alpha}$.

Let first $m = 2k$ (even).

Consider now for $j = 1, \dots, k$. Let

$$\Pi(i)_j^- = \mathcal{W}(p)^{A_m} \cdot e^{\frac{-p-1+i}{2p}\alpha - (j-1)\alpha}, \quad j = 1, \dots, k.$$

$$\Pi(i)_j^+ = \mathcal{W}(p)^{A_m} \cdot Q^{2j-1} e^{\frac{-p-1+i}{2p}\alpha - (j-1)\alpha} \quad j = 1, \dots, k.$$

Similarly, for $m = 2k + 1$ (odd case) we let $\Pi(i)_j^\pm$ as above. In addition, we define

$$\Pi(i)_m = Q^{2k+1} e^{\frac{-p-1+i}{2p}\alpha - k\alpha}.$$

Proposition 4.4. Let $m = 2k$. As Virasoro modules

For $j = 1, \dots, k$

$$\begin{aligned} \Pi(i)_j^- = \Pi(i)_j^+ &= \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=j}^{m-j} L(c_{p,1}, h_{i+p, 2(nm+k)+1}) \\ &+ \bigoplus_{n=0}^{\infty} (2n+2) \bigoplus_{k=m-j+1}^{m+j-1} L(c_{p,1}, h_{i+p, 2(nm+k)+1}). \end{aligned}$$

Proposition 4.5. *Let $m = 2k + 1$. As Virasoro modules
For $j = 1, \dots, k$*

$$\begin{aligned}\Pi(i)_j^- &= \Pi(i)_j^+ = \bigoplus_{n=0}^{\infty} (2n+1) \bigoplus_{k=j}^{m-j} L(c_{p,1}, h_{i+p, 2(nm+k)+1}). \\ &+ \bigoplus_{n=0}^{\infty} (2n+2) \bigoplus_{k=m-j+1}^{m+j-1} L(c_{p,1}, h_{i+p, 2(nm+k)+1}). \\ \Pi(i)_m &= \bigoplus_{n=1}^{\infty} (2n) \bigoplus_{k=0}^{m-1} L(c_{p,1}, h_{i, 2(nm+\frac{m-1}{2}+k)+1}).\end{aligned}$$

As in the proof of Theorem 4.3 we easily see that $\Pi(i)_j^{\pm}$ are all irreducible.

4.3. Irreducible $\mathcal{W}(p)^{A_m}$ -modules: twisted series. Recall a well-known fact.

Lemma 4.6. *Let $m \geq 2$. For $j = 0, \dots, 2p-1$, $i = 1, \dots, m-1$, the space $V_{L+\frac{j-\frac{i}{m}}{2p}\alpha}$ has a σ^i -twisted V_L -module structure. Moreover, $V_{L+\frac{j-\frac{i}{m}}{2p}\alpha}$ is an ordinary $V_L^{A_m}$ (and thus $\mathcal{W}(p)^{A_m}$ -module).*

We immediately get decomposition of $V_{L+\frac{j-\frac{i}{m}}{2p}\alpha}$ into $\mathcal{W}(p)^{A_m}$ -modules:
For $k = 0, \dots, m-1$, we let

$$R(i, j, k) := \bigoplus_{s \in \mathbb{Z}} M(1) \otimes e^{\frac{j-\frac{i}{m}}{2p}\alpha + (ms+k)\alpha}$$

From the Feigin-Fuchs classification of modules, we conclude that each summand appearing in the decomposition is irreducible as Virasoro algebra module.

Thus we get

Corollary 4.7. *Each $R(i, j, k)$ is an irreducible $\mathcal{W}(p)^{A_m}$ -module. All together, twisted V_L -modules yield $2pm(m-1)$ irreducible $\mathcal{W}(p)^{A_m}$ -modules.*

Next result will identify lowest weight vector in $R(i, j, k)$.

Lemma 4.8. *There is a unique $\ell \in \mathbb{Z}$ such that*

$$-(m-1)p \leq \ell \leq (m+1)p-1 \quad \text{and} \quad e^{\frac{\ell-\frac{i}{m}}{2p}\alpha} \in R(i, j, k).$$

Moreover, $e^{\frac{\ell-\frac{i}{m}}{2p}\alpha}$ is a lowest weight vector in $R(i, j, k)$ and

$$L(0)e^{\frac{\ell-\frac{i}{m}}{2p}\alpha} = h_{\ell+1-i/m,1}e^{\frac{\ell-\frac{i}{m}}{2p}\alpha}, \quad H(0)e^{\frac{\ell-\frac{i}{m}}{2p}\alpha} = \left(\ell - \frac{i}{m}\right)e^{\frac{\ell-\frac{i}{m}}{2p}\alpha}.$$

4.4. Irreducible $\mathcal{W}(p)^{A_m}$ -modules: lowest weights. Irreducible modules constructed above are of lowest weight type with respect to $(L(0), H(0))$. The list of the corresponding lowest weights can be found in the following tables:

Let $m = 2k$.

module M	lowest weights	$\dim M(0)$
$\Lambda(i)_0$	$(h_{i,1}, 0)$	1
$\Lambda(i)_j^+$	$(h_{i,2j+1}, \binom{-2jp-1+i}{2p-1})$	1
$\Lambda(i)_j^-$	$(h_{i,2j+1}, -\binom{-2jp-1+i}{2p-1})$	1
$\Lambda(i)_m$	$(h_{i,2k+1}, \binom{-2kp-1+i}{2p-1})$	2
$\Pi(i)_j^+$	$(h_{p+i,2j+1}, \binom{-(2j-1)p-1+i}{2p-1})$	1
$\Pi(i)_j^-$	$(h_{p+i,2j+1}, -\binom{-(2j-1)p-1+i}{2p-1})$	1
$R(i, j, k)$	$(h_{\ell+1-i/m,1}, \binom{\ell-\frac{i}{m}}{2p-1})$	1

The set of lowest conformal weights is

$$S_m := \{h_{i,2j+1} \mid i = 1, \dots, p, j = 0, \dots, k\} \cup \{h_{p+i,2j+1} \mid i = 1, \dots, p, j = 1, \dots, k\} \\ \cup \{h_{\ell+1-i/m,1} \mid p \leq \ell \leq (m+1)p-1, 1 \leq i \leq m-1\}$$

Let $m = 2k+1$.

module M	lowest weights	$\dim M(0)$
$\Lambda(i)_0$	$(h_{i,1}, 0)$	1
$\Lambda(i)_j^+$	$(h_{i,2j+1}, \binom{-2jp-1+i}{2p-1})$	1
$\Lambda(i)_j^-$	$(h_{i,2j+1}, -\binom{-2jp-1+i}{2p-1})$	1
$\Pi(i)_m$	$(h_{p+i,2k+3}, -\binom{-(2k+1)p-1+i}{2p-1})$	2
$\Pi(i)_j^+$	$(h_{p+i,2j+1}, \binom{-(2j-1)p-1+i}{2p-1})$	1
$\Pi(i)_j^-$	$(h_{p+i,2j+1}, -\binom{-(2j-1)p-1+i}{2p-1})$	1
$R(i, j, k)$	$(h_{\ell+1-i/m,1}, \binom{\ell-\frac{i}{m}}{2p-1})$	1

The set of lowest conformal weights is

$$S_m := \{h_{i,2j+1} \mid i = 1, \dots, p, j = 0, \dots, k-1\} \cup \{h_{p+i,2j+1} \mid i = 1, \dots, p, j = 1, \dots, k+1\} \\ \cup \{h_{\ell+1-i/m,1} \mid p \leq \ell \leq (m+1)p-1, 1 \leq i \leq m-1\}$$

Number ℓ is defined as in Lemma 4.8.

Theorem 4.9. *The Λ, Π and R families contain $2m^2p$ non-isomorphic irreducible $\mathcal{W}(p)^{A_m}$ -modules. Lowest weights of irreducible modules are*

$$(x, y) \in \mathbb{C}^2 \text{ such that } x \in S_m, y^2 = C_p P(x).$$

The set of lowest conformal weights S_m has $(m^2+1)p$ elements.

Conjecture 4.10. *The Λ, Π and R families of $\mathcal{W}(p)^{A_m}$ -modules provides a complete list of irreducible $\mathcal{W}(p)^{A_m}$ -modules. In particular, the vertex operator algebra has $2m^2p$ irreducible modules.*

5. MODULES FOR $\mathcal{W}(p)^{A_2}$

In this section we shall give some evidence for the Conjecture 4.10. For simplicity we shall study here the case when $m = 2$, but similar analysis can be made for general m .

Recall that $\mathcal{W}(p)$ is realized as a vertex subalgebra of the lattice vertex algebra V_L , where

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2p.$$

As a vertex algebra $\mathcal{W}(p)^{A_2}$ is generated by

$$F^{(2)} = e^{-2\alpha}, \quad E^{(2)} = Q^4 e^{-2\alpha}, \quad H = Q e^{-\alpha}, \quad \omega,$$

and it is realized as a vertex subalgebra of the lattice vertex algebra $V_{\mathbb{Z}2\alpha}$.

Now we shall recall construction $8p$ -irreducible modules for $\mathcal{W}(p)^{A_2}$ from Section 4. First we shall start with $\mathcal{W}(p)$ -modules:

$$\Lambda(1), \dots, \Lambda(p), \Pi(1), \dots, \Pi(p).$$

Recall that these modules can be constructed as follows:

$$\Lambda(i) = \mathcal{W}(p) e^{\frac{i-1}{2p}\alpha}, \quad i = 1, \dots, p;$$

$$\Pi(i) = \mathcal{W}(p) \cdot e^{\frac{-p-1+i}{2p}\alpha}, \quad i = 1, \dots, p.$$

These modules are \mathbb{Z}_2 -graded and admits the following decomposition into $\mathcal{W}(p)^{A_2}$ -modules:

$$\Lambda(i) = \Lambda(i)_0 \bigoplus \Lambda(i)_2, \quad \Pi(i) = \Pi(i)_1^+ \bigoplus \Pi(i)_1^-,$$

where

$$\Lambda(i)_0 = \mathcal{W}(p)^{A_2} \cdot e^{\frac{i-1}{2p}\alpha}, \quad \Lambda(i)_2 = \mathcal{W}(p)^{A_2} \cdot e^{\frac{i-1}{2p}\alpha - \alpha},$$

and

$$\Pi(i)_1^+ = \mathcal{W}(p)^{A_2} \cdot e^{\frac{-p-1+i}{2p}\alpha}, \quad \Pi(i)_1^- = \mathcal{W}(p)^{A_2} \cdot Q e^{\frac{-p-1+i}{2p}\alpha}.$$

Next we have modules:

$$R(3p-j) = \mathcal{W}(p)^{A_2} \cdot e^{\frac{j-1/2}{2p}\alpha}, \quad j = -p, \dots, 3p-1.$$

Using parametrization from Section 4 we get

$$R(3p-j) = \begin{cases} R(1, 2p+j, 1) & \text{if } -p \leq j \leq -1 \\ R(1, j, 0) & \text{if } 0 \leq j \leq 2p-1 \\ R(1, j-2p, 1) & \text{if } 2p \leq j \leq 3p-1 \end{cases}$$

It is easy to see that all the modules above are irreducible and inequivalent. Moreover, these modules are of lowest weight type with respect to $(L(0), H(0))$. The list of the corresponding lowest weights can be found in

the following table (compared to the previous tables here we used slightly different h -parametrization):

module M	lowest weights	$\dim M(0)$
$\Lambda(i)_0$	$(h_{i,1}, 0)$	1
$\Lambda(i)_2$	$(h_{i,3}, \binom{-2p-1+i}{2p-1})$	2
$\Pi(i)_1^+$	$(h_{3p-i,1}, \binom{-p-1+i}{2p-1})$	1
$\Pi(i)_1^-$	$(h_{3p-i,1}, -\binom{-p-1+i}{2p-1})$	1
$R(j)$	$(h_{3p+1/2-j,1}, \binom{3p-1/2-j}{2p-1})$	1

Conjecture 4.10 says that the set

$$\{\Lambda(i)_0, \Lambda(i)_2, \Pi^\pm(i), R(j), \quad 1 \leq i \leq p, \quad 1 \leq j \leq 4p\}$$

is a complete list of irreducible $\mathcal{W}(p)^{A_2}$ -modules.

5.1. Zhu's algebra for $\mathcal{W}(p)^{A_2}$. We shall present certain results and conjectures for Zhu's algebra for $\mathcal{W}(p)^{A_2}$. By believe that similar results hold for general m , but in the case $m = 2$ it is easier to verify these conjectures by using computational software (e.g. Mathematica/Maple). We shall apply methods developed in [5], [6]. We omit some details which are explained in our previous papers.

Zhu's algebra $A(\mathcal{W}(p)^{A_2})$ is generated by

$$[E^{(2)}], [F^{(2)}], [H], [\omega].$$

Denote $H^{(2)} = Q^2 e^{-2\alpha}$, $F^{(2)} = e^{-2\alpha}$, $E^{(2)} = Q^4 e^{-2\alpha}$. Consider

$$E^{(2)} \circ F^{(2)} \in \overline{M(1)}.$$

We have

$$\begin{aligned}
0 &= Q^4(F^{(2)} \circ F^{(2)}) = E^{(2)} \circ F^{(2)} + F^{(2)} \circ E^{(2)} \\
&\quad + 4(Q^3 e^{-2\alpha} \circ Q e^{-2\alpha} + Q e^{-2\alpha} \circ Q^3 e^{-2\alpha}) + 6H^{(2)} \circ H^{(2)} \\
&= E^{(2)} \circ F^{(2)} + F^{(2)} \circ E^{(2)} + 6H^{(2)} \circ H^{(2)} - 4H^{(2)} \circ H^{(2)} \\
&\quad + 4Q((Q^2 e^{-2\alpha} \circ Q e^{-2\alpha} + Q e^{-2\alpha} \circ Q^2 e^{-2\alpha})) \\
&= E^{(2)} \circ F^{(2)} + F^{(2)} \circ E^{(2)} + 2H^{(2)} \circ H^{(2)} \\
&\quad + 4Q^2(Q e^{-2\alpha} \circ Q e^{-2\alpha}).
\end{aligned}$$

Lemma 5.1.

$$Q^2(Q e^{-2\alpha} \circ Q e^{-2\alpha}) \in O(\overline{M(1)}).$$

Proof. One can easily see that

$$[Q^2(Q e^{-2\alpha} \circ Q e^{-2\alpha})] = F([\omega]) * [H^{(2)}]$$

for certain polynomial $\deg F \leq 3p - 1$. But by construction

$$Q^2(Qe^{-2\alpha} \circ Qe^{-2\alpha}) \in O(\mathcal{W}(p)^{A_2}),$$

so it should act trivially on top components of $\mathcal{W}(p)^{A_2}$ -modules. So we should choose $\mathcal{W}(p)^{A_2}$ -modules on whose top components $H^{(2)}(0)$ does not act trivially. In particular we have:

$$F(h_{i,3}) = 0, F(h_{3p+1/2-j,1}) = 0, \quad i = 1, \dots, p, \quad j = 1, \dots, 2p.$$

Therefore, we have constructed $3p$ zeros of polynomial F . This forces $F = 0$. The proof follows. \square

Lemma implies:

$$E^{(2)} \circ F^{(2)} + F^{(2)} \circ E^{(2)} = -2H^{(2)} \circ H^{(2)} - 4Q^2(Qe^{-2\alpha} \circ Qe^{-2\alpha}) \in O(\overline{M(1)}).$$

Next we notice;

$$E^{(2)} \circ F^{(2)} - F^{(2)} \circ E^{(2)} \in U(Vir)Qe^{-\alpha} + U(Vir)Q^3e^{-3\alpha}.$$

Then by using the structure of Zhu's algebra for $\overline{M(1)}$ [6], we get

$$(2) \quad [E^{(2)} \circ F^{(2)}] = [H] * f_1([\omega]) \in A(\overline{M(1)}), \quad \text{for certain } f_1 \in \mathbb{C}[x],$$

and

$$[E^{(2)} \circ F^{(2)}] = 0 \in A(\mathcal{W}(p)^{A_2}).$$

This yields

$$(3) \quad [H] * \prod_{i=1}^p([\omega] - h_{3p-i,1})([\omega] - h_{i,3}) \prod_{j=1}^{2p}([\omega] - h_{3p+1/2-j,1}) * s([\omega]) = 0$$

where $s(x)$ is a certain polynomial of degree (at most) $p - 1$. This follows from the evaluation of relation (2) on lowest components of $\mathcal{W}(p)^{A_2}$ -modules constructed earlier.

Next we notice that $\Psi(E^{(2)} * F^{(2)} - F^{(2)} * E^{(2)}) = -(E^{(2)} * F^{(2)} - F^{(2)} * E^{(2)})$ which implies that

$$E^{(2)} * F^{(2)} - F^{(2)} * E^{(2)} \in U(Vir)Qe^{-\alpha}.$$

By using the fact that $E^{(2)}(0)$ and $F^{(2)}(0)$ commute on lowest components of modules $\Pi(i)_1^\pm$ and $R(j)$ we get the following formula:

$$(4) \quad \begin{aligned} [[E^{(2)}], [F^{(2)}]] &= [E^{(2)}] * [F^{(2)}] - [F^{(2)}] * [E^{(2)}] \\ &= [H] * \prod_{i=1}^p([\omega] - h_{3p-i,1}) \prod_{j=1}^{2p}([\omega] - h_{3p+1/2-j,1}) * r([\omega]) \end{aligned}$$

for certain polynomial $r(x)$ of degree (at most) $2p - 2$.

Remark 5.2. Since $E^{(2)}(0)$, $F^{(2)}(0)$ and $H(0)$ acts non-trivially on $\mathcal{W}(p)^{A_2}$ -modules $\Lambda(i)_2$, we get that polynomial $r(x)$ is non-trivial.

Since

$$[H \circ F^{(2)}] = f_2([\omega]) * [F^{(2)}]$$

for certain polynomial f_2 , $\deg f_2 = p$, by evaluating this relation on lowest components of modules $\Lambda(i)^-$, we get that

$$f_2(x) = K\ell(x) \quad \ell(x) = \prod_{i=1}^p (x - h_{i,3}).$$

Non-triviality of the constant K can be obtained by the following constant term identity:

Conjecture 5.3. *Let $u = e^{-\alpha} \circ Q^3 e^{-2\alpha} \in \overline{M(1)}$. Then $u(0)$ acts on highest weight vector v_λ of the $M(1)$ -module $M(1, \lambda)$ as follows*

$$\begin{aligned} u(0)v_\lambda &= \text{Res}_{z, z_1, z_2, z_3} \frac{(1+z)^{2p-1-t}(1+z_1)^t(1+z_2)^t(1+z_3)^t}{z^{2+2p}(z_1 z_2 z_3)^{4p}} \\ &\quad \cdot (1 - \frac{z_1}{z})^{-2p} (1 - \frac{z_2}{z})^{-2p} (1 - \frac{z_3}{z})^{-2p} (z_1 - z_2)^{2p} (z_1 - z_3)^{2p} (z_2 - z_3)^{2p} v_\lambda \\ &= A_p \binom{t+2p}{4p-1} \binom{t}{4p-1} v_\lambda \quad (A_p \neq 0). \end{aligned}$$

Since

$$u = 3/2 Q^2 (H \circ F^{(2)}) \in A(\overline{M(1)}),$$

the above conjecture implies non-triviality of constant K .

So assume that above conjecture holds. (We verified this conjecture using Mathematica up to $p \leq 10$.) Then

$$(5) \quad \ell([\omega]) * [F^{(2)}] = 0, \quad \ell(x) = \prod_{i=1}^p (x - h_{i,3}).$$

Similarly,

$$\ell([\omega]) * [E^{(2)}] = 0.$$

Remark 5.4. *Assume that polynomials $r(x)$ and $s(x)$ are relatively prime. Then relations (3)-(5) will imply that $[H] * h([\omega]) = 0$ where*

$$h(x) = \prod_{i=1}^p (x - h_{3p-i,1})(x - h_{i,3}) \prod_{j=1}^{2p} (x - h_{3p+1/2-j,1}) = 0.$$

in Zhu's algebra $A(\mathcal{W}(p)^{A_2})$. This will prove Conjecture 4.10.

In what follows, we will see that Conjecture 4.10 holds for $p \leq 5$. Similar calculations can be made by computer for small values of p . We checked this conjecture up to $p = 10$.

By using Mathematica/Maple we get a list of polynomials $s(x)$ and $r(x)$ for $p \leq 5$ is (up to a scalar factor):

p	$s(x)$	$r(x)$
2	$17x + 28$	$42 - 25x + 10x^2$
3	$6006 + 937x + 932x^2$	$60060 - 47123x + 15897x^2 - 2520x^3 +$ $+336x^4$
4	$1312740 - 8809x + 81758x^2 + 25952x^3$	$5168913750 - 4548646125x + 1727438350x^2 - 360026392x^3 +$ $+45686256x^4 - 3351040x^5 + 225280x^6$
5	$3480248772 - 309156003x + 118443661x^2$ $+24302920x^3 + 6427600x^4$	$63145633719168 - 59216427967788x + 24520167453753x^2$ $-5855373170478x^3 + 890653763025x^4 - 89486430800x^5$ $+6052956000x^6 - 255840000x^7 + 10400000x^8$

They are relatively prime.

Conjecture 5.5.

- (i) *Zhu's algebra $A(\mathcal{W}(p)^{A_2})$ is generated by $[\omega], [H], [E^{(2)}], F^{(2)}$ which satisfy the following relations:*

$$[E^{(2)}]^2 = [F^{(2)}]^2 = 0, [H]^2 = P(x) = C_p \prod_{i=1}^{2p-1} (x - h_{i,1})$$

$$\ell([\omega]) * [F^{(2)}] = \ell([\omega]) * [E^{(2)}] = 0, \quad h([\omega]) * [H] = 0.$$

- (ii) *The center of Zhu's algebra $A(\mathcal{W}(p)^{A_2})$ is isomorphic to*

$$\mathbb{C}[x]/\langle g_{2,p}(x) \rangle$$

where $\langle g_{2,p}(x) \rangle$ is the principal ideal generated by polynomial

$$g_{2,p}(x) = \prod_{i=1}^{3p-1} ([\omega] - h_{i,1}) \prod_{i=1}^p ([\omega] - h_{i,3}) \prod_{i=1}^{2p} ([\omega] - h_{3p+1/2-j,1}).$$

- (iii) *Dimension of $A(\mathcal{W}(p)^{A_2})$ is $12p - 1$.*

Remark 5.6. *From this conjecture will follow that $\mathcal{W}(p)^{A_2}$ does not have "new" logarithmic modules, only logarithmic modules which are obtained by restriction of logarithmic modules for $\mathcal{W}(p)$ (cf. [6]). We believe that similar statement holds for general m . More evidence for this statement will be presented in Section 6*

In the simplest case $p = 2$, we get relations

$$(6) \quad [H] * ([\omega] - 3)([\omega] - \frac{15}{8})([\omega] - 1)([\omega] - \frac{3}{8})$$

$$([\omega] - \frac{45}{32})([\omega] - \frac{21}{32})([\omega] - \frac{5}{32})([\omega] + \frac{3}{32})(17[\omega] + 28) = 0.$$

$$(7) \quad [[E^{(2)}], [F^{(2)}]] = \nu[H] * ([\omega] - 1)([\omega] - \frac{3}{8})$$

$$([\omega] - \frac{45}{32})([\omega] - \frac{21}{32})([\omega] - \frac{5}{32})(42 - 25[\omega] + 10[\omega]^2)$$

By combining (5) and (7) we get

$$(8) \quad [H] * ([\omega] - 3)([\omega] - \frac{15}{8})([\omega] - 1)([\omega] - \frac{3}{8}) \\ ([\omega] - \frac{45}{32})([\omega] - \frac{21}{32})([\omega] - \frac{5}{32})([\omega] + \frac{3}{32})(42 - 25[\omega] + 10[\omega]^2) = 0.$$

Now (6) and (8) implies that

$$(9) \quad [H] * ([\omega] - 3)([\omega] - \frac{15}{8})([\omega] - 1)([\omega] - \frac{3}{8}) \\ ([\omega] - \frac{45}{32})([\omega] - \frac{21}{32})([\omega] - \frac{5}{32})([\omega] + \frac{3}{32}) = 0.$$

Proposition 5.7. *Conjectures 4.10 and 5.5 holds for $m = 2$ and p small ($p \leq 10$).*

Let us describe the structure of algebra $\mathcal{P}(\mathcal{W}(p)^{A_m}) = \mathcal{W}(p)^{A_m} / C_2(\mathcal{W}(p)^{A_m})$ in the case $m = p = 2$. It is generated by $\overline{E^{(2)}}, \overline{F^{(2)}}, \overline{H}, \overline{\omega}$. The following relations holds:

$$\begin{aligned} \overline{\omega}^{12} &= \overline{E^{(2)}}^2 = \overline{F^{(2)}}^2 = 0 \\ \overline{H}^2 &= \nu \overline{\omega}^3, \quad \overline{E^{(2)}F^{(2)}} \in \mathcal{P}(\overline{M(1)}) \\ \overline{\omega}^9 \overline{H} &= \overline{\omega}^2 \overline{E^{(2)}} = \overline{\omega}^2 \overline{F^{(2)}} = 0. \end{aligned}$$

Remark 5.8. *We believe that there are no further relations and therefore $\dim \mathcal{P}(\mathcal{W}(2)^{A_2}) = 25$. So $\mathcal{W}(2)^{A_2}$ is (most likely) new example of vertex operator algebra such that $\dim \mathcal{P}(V) > \dim A(V)$.*

6. IRREDUCIBLE CHARACTERS

In this section we compute the $SL(2, \mathbb{Z})$ -closure of the character of $\mathcal{W}(p)^{A_m}$. Set

$$\begin{aligned} \Theta_{\lambda,k}(\tau) &= \sum_{n \in \mathbb{Z} + \frac{\lambda}{2k}} q^{kn^2} \\ \partial \Theta_{\lambda,k}(\tau) &= \sum_{n \in \mathbb{Z} + \frac{\lambda}{2k}} 2knq^{kn^2} \end{aligned}$$

Define

$$P_{\lambda,k}(\tau) := \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (2n+1) q^{k(n+\frac{\lambda}{2k})^2} = \frac{(k-\lambda)\Theta_{\lambda,k}(\tau) + (\partial\Theta)_{\lambda,k}(\tau)}{k\eta(\tau)},$$

and

$$Q_{\lambda,k}(\tau) := \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (2n) q^{k(n+\frac{\lambda}{2k})^2} = \frac{(-\lambda)\Theta_{\lambda,k}(\tau) + (\partial\Theta)_{\lambda,k}(\tau)}{k\eta(\tau)}.$$

Observe the relations

$$(10) \quad \begin{aligned} Q_{-\lambda,k}(\tau) &= -Q_{\lambda,k}(\tau), \\ Q_{2k+\lambda,k}(\tau) &= Q_{\lambda,k}(\tau) - 2\frac{\Theta_{\lambda,k}}{\eta(\tau)}, \end{aligned}$$

By using decomposition of $\mathcal{W}(p)^{A_m}$ into irreducible Virasoro modules we obtain

$$\begin{aligned} & \text{ch}_{\mathcal{W}(p)^{A_m}}(\tau) \\ &= \frac{1}{\eta(\tau)} \left(\sum_{n \geq 0} (2n+1) q^{p(mn + \frac{p-1}{2p})^2} - \sum_{n=1}^{\infty} (2n-1) q^{p(mn - (m-1) - \frac{p-1}{2p})^2} \right) \\ &+ \frac{1}{\eta(\tau)} \left(\sum_{n \geq 0} (2n+1) q^{p(mn+1 + \frac{p-1}{2p})^2} - \sum_{n=1}^{\infty} (2n-1) q^{p(mn - (m-2) - \frac{p-1}{2p})^2} \right) \\ &+ \cdots + \frac{1}{\eta(\tau)} \left(\sum_{n \geq 0} (2n+1) q^{p(mn+m-1 + \frac{p-1}{2p})^2} - \sum_{n=1}^{\infty} (2n-1) q^{p(mn - \frac{p-1}{2p})^2} \right). \end{aligned}$$

The first and the last term combine into (after the substitution $n \mapsto -n$ in the last term)

$$\frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (2n+1) q^{pm^2(n + \frac{p-1}{2pm})^2} = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (2n+1) q^{pm^2(n + \frac{mp-m}{2pm^2})^2} = P_{mp-m, pm^2}(\tau)$$

Similarly, we combine the remaining terms and obtain

Theorem 6.1. (*Characters of $\Lambda(i)_j^{\pm}$*) For $i = 1, \dots, p$

$$(11) \quad \text{ch}_{\Lambda(i)_0}(\tau) = P_{m(p-i), pm^2}(\tau) + \cdots + P_{m(p-i)+2pm, pm^2}(\tau) + \cdots + P_{m((p-i)+2p(m-1)), pm^2}(\tau).$$

In particular,

$$\text{ch}_{\mathcal{W}(p)^{A_m}}(\tau) = P_{m(p-1), pm^2}(\tau) + P_{m(3p-1), pm^2}(\tau) + \cdots + P_{m(2p(m-1)+(p-1)), pm^2}(\tau)$$

For $j = 1, \dots, k-1$,

$$(12) \quad \begin{aligned} \text{ch}_{\Lambda(i)_j^{\pm}}(\tau) &= P_{2pmj+mi, pm^2}(\tau) + \cdots + P_{mi+2pm(m-2), pm^2}(\tau) \\ &+ Q_{mi-2pm, pm^2}(\tau) + \cdots + Q_{mi+2pm(j-1), pm^2}(\tau) \\ \text{ch}_{\Lambda(i)_m}(\tau) &= Q_{mi-pm^2, pm^2}(\tau) + \cdots + Q_{mi+pm^2-2pm, pm^2}(\tau). \end{aligned}$$

The next lemma follows easily by looking at q -expansion

Lemma 6.2.

$$\frac{1}{m} \sum_{j=0}^{m-1} \partial \Theta_{sm+2pj, pm^2}(\tau) = \partial \Theta_{s,p}(\tau)$$

By using this lemma and formula (10), we see that

$$(13) \quad \begin{aligned} \text{ch}_{\Lambda(i)_j^\pm}(\tau) &= P_{2pmj+mi,pm^2}(\tau) + \cdots + P_{mi+2pm(m-2),pm^2}(\tau) \\ &+ Q_{mi-2pm,pm^2}(\tau) + \cdots + Q_{mi+2pm(j-1),pm^2}(\tau), \end{aligned}$$

can be rewritten as

$$\text{ch}_{\Lambda^\pm(i)_j}(\tau) = \sum_i \lambda_i \frac{\Theta_{im,pm^2}(\tau)}{\eta(\tau)} + \nu \frac{\partial \Theta_{i,p}(\tau)}{\eta(\tau)}$$

for some choice of constants λ_i and $\nu \neq$.

Theorem 6.3. (*Characters of $\Pi(i)_j^\pm$*) For $i = 1, \dots, p$

$$(14) \quad \text{ch}_{\Pi(i)_m}(\tau) = Q_{(p-2m-i)m,pm^2} + \cdots + Q_{(p(m-1)-i)m,pm^2}$$

$$(15) \quad \begin{aligned} \text{ch}_{\Pi(i)_j^\pm}(\tau) &= P_{(2pj-i)m,pm^2} + \cdots + P_{(2p(m-j)-i)m,pm^2} \\ &+ Q_{(2p-2pj-i)m,pm^2} + \cdots + Q_{(2pj-2p-i)m,pm^2}. \end{aligned}$$

Theorem 6.4. (*Characters of $R(i, j, k)$*) We have

$$\text{ch}_{R(i,j,k)}(\tau) = \frac{1}{\eta(\tau)} \sum_{s \in \mathbb{Z}} q^{pm^2(s - \frac{m(p-1-j-2kp)+i}{2m^2p})^2},$$

which also equals

$$\frac{\Theta_{m(p-1-j-2kp)+i,m^2p}}{\eta(\tau)}.$$

Due to symmetry, we have

$$\text{ch}_{R(i,j,k)}(\tau) = \text{ch}_{R(m-i,2p-j-1,m-k)}(\tau).$$

We also have

$$\Theta_{\lambda+2m^2p,m^2p} = \Theta_{\lambda,mp^2} = \Theta_{-\lambda,mp^2} = \Theta_{2m^2p-\lambda,mp^2}.$$

Consider now the span of all twisted characters. By choosing i, j and k in the given range we conclude that this space is spanned by

$$\frac{\Theta_{i,m^2p}}{\eta(\tau)},$$

where i is not divisible by m . The total space of such characters is $m^2p - mp$ -dimensional.

Theorem 6.5. *The modular closure of irreducible modules Λ , Π and R families (conjecturally all irreducible modules) is $m^2p + 2p - 1$ -dimensional spanned by*

$$\frac{\Theta_{i,pm^2}(\tau)}{\eta(\tau)},$$

where $i = 0, \dots, pm^2$,

$$\frac{\partial \Theta_{i,p}(\tau)}{\eta(\tau)},$$

where $i = 1, \dots, p-1$, and

$$\frac{\tau \partial \Theta_{i,p}}{\eta(\tau)},$$

where $i = 1, \dots, p-1$.

Proof. We have already seen that each character of Λ and Π type modules are expressible in terms of $\frac{\Theta_{j,pm^2}(\tau)}{\eta(\tau)}$ and $\frac{\partial \Theta_{i,p}(\tau)}{\eta(\tau)}$, $i = 1, \dots, p-1$. But we already know that the space of characters for the triplet vertex operator algebra $\mathcal{W}(p)$ includes $\frac{\Theta_{i,p}(\tau)}{\eta(\tau)}$ and $\frac{\tau \Theta'_{i,p}(\tau)}{\eta(\tau)}$, so they must be included in the $SL(2, \mathbb{Z})$ closure of $\mathcal{W}(p)^{A_m}$. To finish the proof we only have to argue that each

$$\frac{\Theta_{j,pm^2}(\tau)}{\eta(\tau)},$$

is included in the closure, actually we will show that it lies in the span of irreducible characters. If i is not divisible by m this follows from consideration of twisted modules preceding the theorem. If j is divisible by m , then we observe that

$$\Theta_{im,pm^2}(\tau) = \sum_{n \in \mathbb{Z}} q^{pm^2(n + \frac{im}{2pm^2})^2} = \sum_{n \in \mathbb{Z}} q^{p(mn + \frac{i}{2p})^2}.$$

But the right hand side (when divided by $\eta(\tau)$), is a sum of characters of irreducible $\mathcal{W}(p)^{A_m}$ -modules. Finally, we use the well-known fact that the span of $\Theta_{i,pm^2}(\tau)$ is $pm^2 + 1$ -dimensional. □

Based on the $m = 2$ case and known irreducible $\mathcal{W}(p)^{A_m}$ -modules we expect

Conjecture 6.6.

- (1) *There are $2m^2p$ irreducible $\mathcal{W}(p)^{A_m}$ -modules up to isomorphism.*
- (2) *The space of one-point functions on the torus is $m^2p + 2p - 1$ -dimensional.*

7. APPENDIX: A CONSTRUCTION OF THE AUTOMORPHISM Ψ OF $\mathcal{W}(p)$

Define the following singular vectors in $\mathcal{W}(p)$:

$$v_{n,i}^{\pm} = (2n-i)! Q^i e^{-n\alpha} \pm (-1)^i! Q^{2n-i} e^{-n\alpha}$$

where $n \in \mathbb{N}$, $i = 0, \dots, n$.

Let $\langle v_{n,i}^{\pm} \rangle$ denotes the irreducible Virasoro submodule generated by $v_{n,i}^{\pm}$. Define the following \mathbb{Z}_2 -graduation on $\mathcal{W}(p)$:

$$\begin{aligned} \mathcal{W}(p)^+ &= \bigoplus_{n=0}^{\infty} \bigoplus_{i=0}^n \langle v_{n,i}^+ \rangle \\ \mathcal{W}(p)^- &= \bigoplus_{n=0}^{\infty} \bigoplus_{i=0}^n \langle v_{n,i}^- \rangle \end{aligned}$$

Clearly,

$$\mathcal{W}(p) = \mathcal{W}(p)^+ \bigoplus \mathcal{W}(p)^-.$$

Let $\Phi \in \text{End}(\mathcal{W}(p))$ such that

$$\Phi|_{\mathcal{W}(p)^\pm} = \pm \text{Id}.$$

Let $H = Qe^{-\alpha}$, $U^- = v_{1,0}^- = 2F - E$, $U^+ = 2F + E$.

Note that $\mathcal{W}(p)$ is strongly generated by ω, H, U^\pm , and that $\omega, U^+ \in \mathcal{W}(p)^+$ and $H, U^- \in \mathcal{W}(p)^-$.

Proposition 7.1. Φ is an automorphism of $\mathcal{W}(p)$:

$$\Phi \in \text{Aut}(\mathcal{W}(p)).$$

Proof. It suffices to check that the generators satisfy:

$$H_j \mathcal{W}(p)^\pm \subset \mathcal{W}(p)^\mp, U_j^\pm \mathcal{W}(p)^\pm \subset \mathcal{W}(p)^\pm, U_j^\mp \mathcal{W}(p)^\pm \subset \mathcal{W}(p)^\mp$$

for every $j \in \mathbb{Z}$. This will follow from Lemmas 7.2, 7.3 and 7.4 below. \square

7.1. Some lemmas.

Lemma 7.2. Let $j \in \mathbb{Z}$. Then

$$H_j v_{n,i}^\pm \in \langle v_{n-1,i-1}^\mp \rangle + \langle v_{n,i}^\mp \rangle + \langle v_{n+1,i+1}^\mp \rangle.$$

Proof. First we notice that

$$H_j Q^i e^{-n\alpha} = Q^i (H_j e^{-n\alpha}) - i Q^{i-1} (E_j e^{-n\alpha}),$$

$$H_j Q^{2n-i} e^{-n\alpha} = Q^{2n-i} (H_j e^{-n\alpha}) - (2n-i) Q^{2n-i-1} (E_j e^{-n\alpha}).$$

Let $\overline{u}_i \in U(\text{Vir})$, $i = -1, 0, 1$ such that

$$E_j e^{-n\alpha} = \overline{u}_{-1} e^{-(n-1)\alpha} + \overline{u}_0 Q e^{-n\alpha} + \overline{u}_1 Q^2 e^{-(n+1)\alpha}.$$

Let \overline{u}_i' , $i = 0, 1$ such that

$$H_j e^{-n\alpha} = \overline{u}_0' e^{-n\alpha} + \overline{u}_1' Q e^{-(n+1)\alpha}.$$

Now we shall find relations between \overline{u}_i' and \overline{u}_i .

By applying Q^n on the previous relation we get:

$$\begin{aligned} & \overline{u}_0' Q^n e^{-n\alpha} + \overline{u}_1' Q^{n+1} e^{-(n+1)\alpha} \\ = & Q^n (H_j e^{-n\alpha}) = n E_j Q^{n-1} e^{-n\alpha} + H_j Q^n e^{-n\alpha} \\ = & n (\overline{u}_{-1} Q^{n-1} e^{-(n-1)\alpha} + \overline{u}_0 Q^n e^{-n\alpha} + \overline{u}_1 Q^{n+1} e^{-(n+1)\alpha}) + H_j Q^n e^{-n\alpha} \end{aligned}$$

Since $H_j Q^n e^{-n\alpha}$ does not contain component inside $\langle Q^n e^{-n\alpha} \rangle$ (cf. [4]) we conclude

$$\overline{u}_0' Q^n e^{-n\alpha} = n \overline{u}_0 Q^n e^{-n\alpha}.$$

This proves that

$$\overline{u}_0' Q^i e^{-n\alpha} = n \overline{u}_0 Q^i e^{-n\alpha} \quad i = 0, \dots, 2n.$$

Similarly,

$$\overline{u_1}' Q^{2n+2} e^{-(2n+2)\alpha} = Q^{2n+1} H_j e^{-n\alpha} = (2n+1) E_j Q^{2n} e^{-n\alpha} = (2n+1) \overline{u_1} Q^{2n+2} e^{-(2n+2)\alpha}$$

This implies

$$\overline{u_1}' Q^i e^{-(n+1)\alpha} = (2n+1) \overline{u_1} Q^i e^{-(n+1)\alpha} \quad i = 0, \dots, 2n+2.$$

We get:

$$\begin{aligned} H_j v_{n,i}^\pm &= (2n-i)! H_j Q^i e^{-n\alpha} \pm (-1)^i i! H_j Q^{2n-i} e^{-n\alpha} \\ &= \overline{u_{-1}} \left((-i)(2n-i)! Q^{i-1} e^{-(n-1)\alpha} \pm (-1)^i i! (-(2n-i)) Q^{2n-i-1} e^{-(n-1)\alpha} \right) \\ &\quad + \overline{u_0} (n-i) \left((2n-i)! Q^i e^{-n\alpha} \mp (-1)^i i! Q^{2n-i} e^{-n\alpha} \right) \\ &\quad + \overline{u_1} \left((2n+1-i)! Q^{i+1} e^{-(n+1)\alpha} \pm (-1)^i (i+1)! Q^{2n+1-i} e^{-(n+1)\alpha} \right) \\ &= -(2n-i) i \overline{u_{-1}} v_{n-1,i-1}^\mp + (n-i) \overline{u_0} v_{n,i}^\mp + \overline{u_1} v_{n+1,i+1}^\mp. \end{aligned}$$

The Lemma holds. \square

Lemma 7.3. *Let $j \in \mathbb{Z}$. Then*

$$U_j^+ v_{n,i}^\pm \in \langle v_{n-1,i-2}^\pm \rangle + \langle v_{n-1,i}^\pm \rangle + \langle v_{n,i-1}^\pm \rangle + \langle v_{n,i+1}^\pm \rangle + \langle v_{n+1,i}^\pm \rangle + \langle v_{n+1,i+2}^\pm \rangle.$$

Proof. First we notice that

$$\begin{aligned} e_j^{-\alpha} Q^i e^{-n\alpha} &= Q^i (e_j^{-\alpha} e^{-n\alpha}) - i Q^{i-1} (H_j e^{-n\alpha}) + \frac{i(i-1)}{2} Q^{i-2} E_j e^{-n\alpha}, \\ e_j^{-\alpha} Q^{2n-i} e^{-n\alpha} &= Q^{2n-i} (e_j^{-\alpha} e^{-n\alpha}) - (2n-i) Q^{2n-i-1} (H_j e^{-n\alpha}) + \\ &\quad + \frac{(2n-i)(2n-i-1)}{2} Q^{2n-i-2} E_j e^{-n\alpha}. \end{aligned}$$

As in the previous Lemma, we define $\overline{u_i}, \overline{u_i}', \overline{u_i}''$ by

$$\begin{aligned} E_j e^{-n\alpha} &= \overline{u_{-1}} e^{-(n-1)\alpha} + \overline{u_0} Q e^{-n\alpha} + \overline{u_1} Q^2 e^{-(n+1)\alpha}, \\ H_j e^{-n\alpha} &= \overline{u_0}' e^{-n\alpha} + \overline{u_1}' Q e^{-(n+1)\alpha}, \\ e_j^{-\alpha} e^{-n\alpha} &= \overline{u_1}'' e^{-(n+1)\alpha}, \end{aligned}$$

By the proof of Lemma 7.2 we get:

$$\begin{aligned} \overline{u_0}' Q^i e^{-n\alpha} &= n \overline{u_0} Q^i e^{-n\alpha}, \\ \overline{u_1}' Q^i e^{-(n+1)\alpha} &= (2n+1) \overline{u_1} Q^i e^{-(n+1)\alpha}, \\ \overline{u_1}'' Q^i e^{-(n+1)\alpha} &= (2n+1)(n+1) \overline{u_1} Q^i e^{-(n+1)\alpha}. \end{aligned}$$

Now we have

$$(16) \quad U_j^+ v_{n,i}^\pm =$$

$$(17) \quad 2(2n-i)! e_j^{-\alpha} Q^i e^{-n\alpha} \pm (-1)^i i! E_j Q^{2n-i} e^{-n\alpha}$$

$$(18) \quad + (2n-i)! E_j Q^i e^{-n\alpha} \pm 2(-1)^i i! e_j^{-\alpha} Q^{2n-i} e^{-n\alpha}.$$

By direct calculation we see that the expression (17) is equal to

$$\begin{aligned}
& i(i-1)\overline{u_{-1}} \left((2n-i)!Q^{i-2}e^{-(n-1)\alpha} \pm (-1)^{i-2}(i-2)!Q^{2n-i}e^{-(n-1)\alpha} \right) + \\
& -i\overline{u_0} \left((2n+1-i)!Q^{i-1}e^{-n\alpha} \pm (-1)^{i-1}(i-1)!Q^{2n+1-i}e^{-n\alpha} \right) + \\
& \overline{u_1} \left((2n+2-i)!Q^i e^{-(n+1)\alpha} \pm (-1)^i i!Q^{2n+2-i}e^{-n\alpha} \right) + \\
= & i(i-1)\overline{u_{-1}}v_{n-1,i-2}^\pm - i\overline{u_0}v_{n,i-1}^\pm + \overline{u_1}v_{n+1,i}^\pm.
\end{aligned}$$

Similarly, the expression (18) is equal to

$$\left((2n-i)(2n-i-1)\overline{u_{-1}} v_{n-1,i}^\pm - (2n-i)\overline{u_0} v_{n,i+1}^\pm + \overline{u_1}v_{n+1,i+2}^\pm \right).$$

The Lemma holds. \square

The proof of the following Lemma is completely analogous to the previous case.

Lemma 7.4. *Let $j \in \mathbb{Z}$. Then*

$$U_j^- v_{n,i}^\pm \in \langle v_{n-1,i-2}^\mp \rangle + \langle v_{n-1,i}^\mp \rangle + \langle v_{n,i-1}^\mp \rangle + \langle v_{n,i+1}^\mp \rangle + \langle v_{n+1,i}^\mp \rangle + \langle v_{n+1,i+2}^\mp \rangle.$$

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